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Group properties of generalized quasi-linear wave equations

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ABSTRACT

In this paper, complete group classification of a class of $(1+1)$-dimensional generalized quasi-linear wave equations is performed by using the Lie–Ovsiannikov method, additional equivalent transformation and furcate split method. Lie reductions of some truly 'variable coefficient' wave equations which are singled out from the classification results are investigated. Some classes of exact solutions of these 'variable coefficient' wave equations are constructed by means of both the reductions and the additional equivalent transformations. The nonclassical symmetries to the generalized quasi-linear wave equation are also studied. This enabled to obtain some exact solutions of the wave equations which are invariant under certain conditional symmetries.

1. Introduction

In this paper we study the properties of Lie symmetry group and Lie symmetry algebra associated with a class of generalized quasi-linear wave equations of the form

$$u_{tt} = \big(H(u)u_x\big)_x + k(x)u, \tag{1}$$

in order to characterize some invariant solutions by means of infinitesimal groups of transformations. Here $H_u \neq 0$, H and k are arbitrary functions of their argument, t is the time coordinate and x is the one-space coordinate.

Eq. (1) is an unified form of many physical examples which is often used to model a wide variety of phenomena in Mechanics and Engineering. For example, in the case $k(x) = 0$, Eq. (1) can be used a model to describe: (a) the flow of one-dimensional gas, (b) shallow water waves theory, (c) longitudinal wave propagation on a moving threadline, (d) dynamics of a finite nonlinear string, (e) elastic-plastic materials, and (f) electromagnetic transmission line (see [2, pp. 50–52] and [3]).

In the case $k(x)$ nonconstant and $H(u)$ constant, Eq. (1) is a special case of the class of equations

$$\rho(x)u_{tt} - \big(p(x)u_x\big)_x = \mu f(x,t,u), \tag{2}$$

which describes the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in non-isotropic media [7,8,18]. Here $\rho(x) > 0$ is the mass per unit length, $p(x) > 0$ is the modulus of elasticity multiplied by the cross-sectional area (see [18, p. 291]), $\mu > 0$ is a parameter, and the nonlinear forcing term $f(x,t,u)$ is $(2\pi/\omega)$-periodic in time. The periodic solutions for Eq. (2) and its special cases (i.e. when $\rho(x) = p(x) \equiv 1$) has been widely investigated since the 1960s (see [8] and the references cited therein).

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Eq. (1) is also a special case of the class of nonlinear wave equation with variable speed and external force

$$u_{tt} = A(x)\big(H(u)u_x\big)_x + B(x)Q(u). \tag{3}$$

When $A(x) = B(x) = 1$, Estévez and Qu obtained a classification of the functional separation of variables solutions by generalized conditional symmetry approach [20]. Recently, the classification has been extended to the whole class of Eq. (3) by group foliation method [27].

The problem of the investigation of group properties [11,23,31,47,49] for the degenerate case $k = 0$ of Eq. (1) (i.e. the class of nonlinear one-dimensional wave equations) was first solved by Ames et al. [3] in 1981. From that well-known paper, the search for symmetries of various kinds of one-dimensional nonlinear wave equations has been considered in many papers in the last two decades [4,9,11,12,15,19,24,28,31,32,40,48,53–57,59]. Although a wide range of wave equations was investigated within the symmetry framework, even 'simplified' $(1 + 1)$-dimensional nonlinear wave models are fraught with a great many of 'symmetry mysteries' which remain to be solved [31]. In particular, the complete group classifications of Eqs. (1)–(3) also remain open (see [28,40] for a detailed literature review about the group classifications of nonlinear wave equations). Therefore, the study of Eqs. (1)–(3) is stimulated not only its physical importance, but also their intrinsic theoretical interest.

Recently, we have presented a preliminary group classifications of Eq. (1) without proof and constructed some exact solutions for certain special classification model [29,30], which can be seen as a start point for the complete classification of Eqs. (1)–(3). In this paper, our research is mainly concentrated on rigorous and exhaustive group classification of the whole class (1) with a detailed proof and Lie symmetry reduction of some truly nonlinear 'variable-coefficient' wave equation (equation with $H_u k_x \neq 0$) from this class. The method used here is a combination of the Lie–Ovsiannikov method [1,49], additional equivalent transformation and furcate split method which was present in recent paper [43,52] and had been applied to investigating a number of different group classification problem [13,28,34,35,51,58,59]. Nonclassical symmetries of the class of generalized quasi-linear wave equations (1) are also investigated. This enabled to obtain some exact solutions of wave equations which are invariant under certain conditional symmetries. The complete group classifications of Eqs. (2) and (3) will be investigated in a subsequent work.

The structure of the paper is the following. In Section 2, classification results on generalized quasi-linear wave equations (1) are presented by means of the Lie–Ovsiannikov method, additional equivalent transformation and furcate split method. Section 3 contains Lie reductions and exact solutions of some truly nonlinear 'variable-coefficient' wave equation which are singled out from the wave equations (1). Analysis of the nonclassical symmetries of the class of variable coefficient nonlinear wave equations (1) is given in Section 4. Finally, some conclusion and discussion are given in Section 5.

2. Group classification of generalized wave equations

Group classification of class (1) is performed in the framework of the Lie–Ovsiannikov method, additional equivalent transformation and furcate split method [1,28,34,43,49,52]. All necessary objects (the equivalence group, principal group, the kernel and all inequivalent extensions of maximal Lie invariance algebras) are found. Moreover, we extend the classical approach with additional equivalence transformations for simplification of the classification results.

According to the algorithm [28,43,49,52] we seek an infinitesimal operator in the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{4}$$

which corresponds to a one-parameter Lie group of local transformations and keeps Eq. (1) invariant. The classical infinitesimal Lie invariance criterion for Eq. (1) to be invariant with respect to the operator (4) read as

$$\mathrm{pr}^{(2)}\,Q(\triangle)\,|_{\triangle=0} = 0, \qquad \triangle = u_{tt} - \big(H(u)u_x\big)_x - k(x)u. \tag{5}$$

Here $\mathrm{pr}^{(2)}\,Q$ is the usual second order prolongation [47,49] of the operator (4). Equating coefficients of linearly independent terms of invariance criterion (5) to zero yields an overdetermined system of linear PDEs (called determining equations). Solving this system we obtain the following assertion.

Proposition 1. *The symmetry group of the generalized quasi-linear wave equations* (1) *for two arbitrary* (*fixed*) *function H and k is generated by the infinitesimal operators of the form*

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + \eta(t,x,u)\partial_u$$

with

$$\eta = \eta^1(t,x)u + \eta^0(t,x), \quad \eta^1(t,x) = \frac{1}{2}\tau_t + \alpha(x), \tag{6}$$

where the real-valued functions $\tau(t)$, $\xi(x)$ and $\eta(t,x,u)$ satisfy the determining equations

$$2(\xi_x - \tau_t) = \frac{H_u}{H}\eta, \tag{7}$$

$$H\xi_{xx} - 2H_u\eta_x - 2H\eta_{xu} = 0, \tag{8}$$

$$\eta_{tt} - H\eta_{xx} - k\eta - (\xi k_x + 2\tau_t k - \eta_u k)u = 0. \tag{9}$$

Thus, group classification of (1) reduces to solving classifying conditions (7)–(9).

Splitting system (7)–(9) with respect to the arbitrary elements and their non-vanishing derivatives gives the equations $\tau_t = 0$, $\xi = 0$, $\eta = 0$ for the coefficients of the operators from $A^{\ker}$ of (1). As a result, the following theorem is true.

Theorem 1. *The Lie algebra of the kernel of principal groups of* (1) *is an one-dimensional algebra* $A^{\ker} = \langle \partial_t \rangle$.

In what follows, we shall carry out the group classification of class (1) up to equivalence under the equivalence group (which we denote by $G^{\sim}$) of the equation under consideration. The equivalence group $G^{\sim}$ consists of equivalence transformations which are nondegenerate change of the variables t, x and u taking any equation of the form (1) into an equation of the same form, generally speaking, with different $H(u)$ and $k(x)$. To find the connected component of the unity of $G^{\sim}$, we have to investigate Lie symmetries of the system that consists of Eq. (1) and some additional conditions, that is to say we must seek for an operator of the Lie algebra $A^{\sim}$ of $G^{\sim}$ in the form

$$X = \tau\partial_t + \xi\partial_x + \eta\partial_u + \rho\partial_H + \varphi\partial_k \tag{10}$$

from the invariance criterion of (1) applied to the system:

$$u_{tt} = \big(H(u)u_x\big)_x + k(x)u_x,$$
$$H_t = H_x = 0, \qquad k_t = k_u = 0. \tag{11}$$

Here u, H and k are considered as differential variables: u on the space (t, x) and H, k on the extended space (t, x, u). The coordinates τ, ξ, η of the operator (10) are sought as functions of t, x, u while the coordinates ρ, φ are sought as functions of t, x, u, H, k.

The invariance criterion of system (11) yields the following determining equations for τ, ξ, η, ρ and φ:

$$\tau_x = \tau_u = \xi_t = \xi_u = \eta_t = \eta_x = \eta_{uu} = 0, \qquad 2\eta_{tu} - \tau_{tt} = 0,$$
$$\rho_t = \rho_x = \rho_k = 0, \qquad \varphi_t = \varphi_u = \varphi_H = 0,$$
$$-2H\eta_{xu} - 2H_u\eta_x + H\xi_{xx} = 0,$$
$$2(\xi_x - \tau_t)H_u - H_u\rho_H - \rho_u = 0,$$
$$2(\xi_x - \tau_t)H - \rho = 0,$$
$$\eta_{tt} - H\eta_{xx} - k\eta - (\varphi + 2k\tau_t - k\eta_u)u = 0. \tag{12}$$

After easy calculations we find from (12):

Proposition 2. *For any* $H(u)$, $k(x)$ *satisfying the system* (12), *Eqs.* (11) *admit the Lie symmetry* (10) *with*

$$\tau = c_1 + c_2 t, \qquad \xi = c_3 x + c_4, \qquad \eta = c_5 u,$$
$$\rho = 2(c_3 - c_1)H, \qquad \varphi = -2c_1 k,$$

where $c_1, \dots, c_5$ *are arbitrary constants.*

Thus, we obtain the following statement.

Theorem 2. *The Lie algebra of the equivalence group* $G^{\sim}$ *for class* (1) *is*

$$A^{\sim} = \langle \partial_t, \partial_x, u\partial_u, x\partial_x + 2H\partial_H, t\partial_t - 2H\partial_H - 2k\partial_k \rangle.$$

Proposition 3. *The equivalence group* $G^{\sim}$ *of class* (1) *is formed by the transformations*

$$\tilde{t} = t\epsilon_1 + \epsilon_2, \qquad \tilde{x} = x\epsilon_3 + \epsilon_4, \qquad \tilde{u} = \epsilon_5 u, \qquad \tilde{H} = H\epsilon_1^{-2}\epsilon_3^2, \qquad \tilde{k} = k\epsilon_1^{-2},$$

where $\epsilon_1, \dots, \epsilon_5$ *are arbitrary constants,* $\epsilon_1\epsilon_3\epsilon_5 \neq 0$. *The connected component of the unity in* $G^{\sim}$ *is formed by continuous transformations having* $\epsilon_1 > 0$, $\epsilon_3 > 0$ *and* $\epsilon_5 > 0$.

Table 1
Results of group classification.

N	$H(u)$	$k(x)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	∂_t
2	$\forall$	1	∂_t, ∂_x
3	$\forall$	x^{-2}	∂_t, $t\partial_t + x\partial_x$
4	u^μ	ϵx^q	∂_t, $\mu qt\partial_t - 2\mu x\partial_x - 2(q+2)u\partial_u$
5	u^μ	ϵe^x	∂_t, $\mu t\partial_t - 2\mu\partial_x - 2u\partial_u$
6	$u^{-\frac{4}{3}}$	$k^1(x)$	∂_t, $-2qt\partial_t + 4(x^2+p)\partial_x - 3(4x+q)u\partial_u$
7	$\forall$	0	∂_t, ∂_x, $t\partial_t + x\partial_x$
8	e^u	0	∂_t, ∂_x, $t\partial_t + x\partial_x$, $x\partial_x + 2\partial_u$
9	u^μ, $\mu \neq -4, -\frac{4}{3}$	ϵ	∂_t, ∂_x, $\mu x\partial_x + 2u\partial_u$
10	$(u+\alpha)^\mu$, $\mu \neq -4, -\frac{4}{3}$	0	∂_t, ∂_x, $t\partial_t + x\partial_x$, $\mu x\partial_x + 2(u+\alpha)\partial_u$
11	$u^{-\frac{4}{3}}$	ϵ	∂_t, ∂_x, $2x\partial_x - 3u\partial_u$, $x^2\partial_x - 3xu\partial_u$
12	$(u+\alpha)^{-\frac{4}{3}}$	0	∂_t, ∂_x, $2t\partial_t + 3(u+\alpha)\partial_u$, $2x\partial_x - 3(u+\alpha)\partial_u$, $x^2\partial_x - 3x(u+\alpha)\partial_u$
13	$(u+\alpha)^{-4}$	0	∂_t, ∂_x, $2t\partial_t + (u+\alpha)\partial_u$, $2x\partial_x - (u+\alpha)\partial_u$, $t^2\partial_t + t(u+\alpha)\partial_u$
14	u^{-4}	1	∂_t, ∂_x, $e^{2t}(\partial_t + u\partial_u)$, $2x\partial_x - u\partial_u$, $e^{-2t}(\partial_t - u\partial_u)$
15	u^{-4}	-1	∂_t, ∂_x, $\cos(2t)\partial_t - \sin(2t)u\partial_u$, $2x\partial_x - u\partial_u$, $\sin(2t)\partial_t + \cos(2t)u\partial_u$

Note. Here $k^1(x) = \epsilon\exp[\int \frac{q}{x^2+p}]$; $p \in \{-1, 0, 1\}$, $\epsilon = \pm 1$, $\alpha \in \{0, 1\}$ mod $G^\sim$; $\mu, q \neq 0$.

Additional equivalence transformations:

1. $6|_{p=-1} \to 4|_{\tilde\mu=-4/3, \tilde q=q/2}$: $\tilde t = t$, $\tilde x = \frac{x-1}{x+1}$, $\tilde u = 2^{-3/2}(x+1)^3 u$;
2. $6|_{p=0} \to 5|_{\tilde\mu=-4/3}$: $\tilde t = t$, $\tilde x = x^{-1}$, $\tilde u = x^3 u$;
3. $10|_{\alpha\neq 0} \to 10|_{\alpha=0}$ $(\mu = -\frac{4}{3})$, $12|_{\alpha\neq 0} \to 12|_{\alpha=0}$, $13|_{\alpha\neq 0} \to 13|_{\alpha=0}$: $\tilde t = t$, $\tilde x = x$, $\tilde u = u + \alpha$;
4. $14 \to 13|_{\alpha=0}$: $\tilde t = -\frac{1}{2}e^{-2t}$, $\tilde x = x$, $\tilde u = e^{-t}u$;
5. $15 \to 13|_{\alpha=0}$: $\tilde t = -\frac{1}{2}e^{-2it}$, $\tilde x = x$, $\tilde u = e^{-it}u$.

Remark 1. There also exists a non-trivial group of discrete equivalence transformations for class (1) generated by three involutive transformations of alternating sign in the sets $\{t, H, k\}$, $\{x\}$ and $\{u\}$. To find the complete discrete equivalence group is very complicated, we should use the direct method. A problem of this sort was first investigated for wave equations by Kingston and Sophocleous (see e.g., [36,37,55]). We will try to discuss the structure of the complete set of discrete equivalence transformation of class (1) in a sequel paper.

Under the transformations in Proposition 3 we can present the following classification results:

Theorem 3. *A complete set of $G^\sim$-inequivalent extensions of $A^{\max} \neq A^{\ker}$ for Eq.* (1) *is exhausted by ones given in Table* 1.

Proof. To prove the theorem we use the furcate split method [28,34,43,52]. The basic idea of this method is based on the fact that the substitution of the coefficients of any operator from $A^{\max} \setminus A^{\ker}$ into the classifying equations results in nonidentity equations for arbitrary elements (see [28,34,43,52] for more details and exhaustive examples of applications). In our case the procedure of looking for the possible cases mostly depends on Eq. (7). For any symmetry operator Eq. (7) gives some equations on H of the general form $(au+b)H_u = cH$, where a, b, c are constants. For all operators from $A^{\max}$ the number l of such independent equations is not greater than 2; otherwise they form an incompatible system on H. l is an invariant value for the transformations from $G^\sim$. Therefore, there exist three inequivalent cases for the value of l: (i) $l=0$: $H(u)$ is arbitrary; (ii) $l=1$: $H(u) = e^u$ or $H(u) = (u+\alpha)^\mu$ ($\alpha = 0$ or $1, \mu \neq 0$) mod $G^\sim$, and (iii) $l=2$: $H(u) = 1$ mod $G^\sim$. Let us consider in more detail the case $H(u) = (u+\alpha)^\mu$ with $\alpha = 0$, i.e., $H(u) = u^\mu$, the other cases can be investigated in a similar way. We attempted to present our calculations in reasonable detail so that verification would be feasible. For this case Eqs. (7) and (6) imply $\eta^0(t,x) = 0$, i.e. $\eta = (\frac{1}{2}\tau_t + \alpha(x))u = \eta^1(t,x)u$. Therefore, Eqs. (7)–(9) can be written as

$$2(\xi_x - \tau_t) = \mu\eta^1, \tag{13}$$

$$\xi_{xx} - 2(\mu+1)\eta^1_x = 0, \tag{14}$$

$$\eta^1_{xx} = 0, \qquad \eta^1_{tt} - \xi k_x - 2\tau_t k = 0. \tag{15}$$

Differentiating Eq. (13) with respect to t and x respectively and noting $\eta^1(t,x) = \frac{1}{2}\tau_t + \alpha(x)$ and Eq. (14) yield equations $(\mu+4)\tau_{tt} = 0$, $(\mu+\frac{4}{3})\eta^1_x = 0$. They are two classifying ones. Thus there are three cases should be considered: $\tau_{tt} = 0$, $\eta^1_x = 0$ if the value $\mu \neq -4, -\frac{4}{3}$, $\eta^1_x = 0$ if the value $\mu = -4$ and $\tau_{tt} = 0$ if the value $\mu = -\frac{4}{3}$.

To the first case, let $\tau = c_2 t + c_1$. Then Eqs. (13) and (14) imply that $\eta^1_t = 0$ and $\xi = c_3 + c_4 x$ respectively. Substituting them into Eq. (15) and solving it under the equivalent group $G^\sim$ we can get $k \in \{\epsilon x^q, \epsilon x^q, \epsilon, 0\}$ mod $G^\sim$, where $\epsilon = \pm 1$. Thus cases 4, 5 with $\mu \neq -4, -\frac{4}{3}$, cases 9 and 10 with $\alpha = 0$ are obtained.

To the second case, from Eq. (15) we can get $k = \frac{\eta^1_{ttt}}{2\tau_{tt}}$ if $\tau_{tt} \neq 0$, which implies that k must take the values $\epsilon = \pm 1$ or 0 mod $G^{\sim}$. These are cases 13, 14 and 15. If $\tau_{tt} = 0$, then we can get cases 4, 5 with $\mu = -4$.

To the third case, from Eq. (13) we know that $\eta^1_t = 0$ because of $\tau_{tt} = 0$. Therefore, if $\eta^1_x \neq 0$, (15) implies that $k \in \{\epsilon \exp[\int \frac{q}{x^2+p}\,dx], \epsilon, 0\}$ mod $G^{\sim}$, where $\epsilon = \pm 1$, $p \in \{-1, 0, 1\}$. When $k = \epsilon \exp[\int \frac{q}{x^2+p}\,dx]$, solving Eqs. (13)–(15) we obtain $\tau = c_1 - \frac{1}{2}c_2 qt$, $\xi = c_2(x^2+p)$, $\eta^1 = -\frac{3}{4}(4x+q)c_2$, which corresponding to case 6. When $k = 0$, from Eqs. (13)–(15) we can get $\tau = c_1 + c_3 t$, $\xi = c_2 + (c_3 + 2c_4)x + c_5 x^2$, $\eta^1 = -3c_4 - 3c_5 x$. Thus case 12 with $\alpha = 0$ are obtained. Finally $k = \epsilon$ corresponding to case 11. If $\eta^1_x = 0$, then we can get cases 4, 5 with $\mu = -\frac{4}{3}$.

The rest of the cases of values H can be studied in an analogous way. □

Remark 2. The parameter function $k^1(x)$ equals to the following functions depending on values of p:

$$p = -1:\quad k^1(x) = \epsilon \left|\frac{x-1}{x+1}\right|^{\frac{q}{2}}; \qquad p = 0:\quad k^1(x) = \epsilon e^{-\frac{q}{x}}; \qquad p = 1:\quad k^1(x) = \epsilon e^{q \arctan x}.$$

Additionally we can assume $q = -1$ mod $G^{\sim}$ if $p = 0$.

Remark 3. Some cases from Table 1 are equivalent with respect to point transformations which obviously do not belong to $G^{\sim}$. These transformations are called *additional equivalence transformations* and lead to simplification of further application of group classification results (see reference [28,52] for details). The simplest way to find such additional equivalences between previously classified equations is based on the fact that equivalent equations have equivalent maximal Lie invariance algebras. Explicit formulas for additional transformations that do not change the value of $H(u)$ are adduced after the tables. Besides these transformations there exist additional point transformations changing $H(u)$ [28]. Thus, e.g., $\tilde{t} = x$, $\tilde{x} = t$, $\tilde{u} = \ln u$ maps case 10 ($\alpha = 0$) to 8. One more example of similar transformations is $\tilde{t} = x$, $\tilde{x} = t$, $\tilde{u} = u^{\mu+1}$, $\tilde{\mu} = -\mu/(\mu+1)$ between equations of form $u_{tt} = (u^{\mu}u_x)_x$, $\mu \neq -1$. In particular, it connects cases 12 and 13. The same transformation applied is a discrete symmetry for equation with $\mu = -2$. The latter two transformations are, indeed, partial cases of more general transformation

$$\tilde{t} = x, \qquad \tilde{x} = t, \qquad \tilde{u} = \int H(u)\,du \tag{16}$$

between equations from class

$$u_{tt} = \big(H(u)u_x\big)_x, \tag{17}$$

where the new transformed value of arbitrary element $\tilde{H}$ is the derivative to the inverse function $u = \hat{H}(\tilde{u})$ for $\tilde{u} = \int H(u)\,du$ [31]. Transformation (16) is nonlocal with respect to the arbitrary element $H(u)$ and therefore can be considered as *generalized extended equivalence transformation* [33,42] in class of nonlinear wave equations (17). One can check that there exist no other point transformations between the equations from Table 1. Using this we can formulate the following theorem.

Theorem 4. *Up to point transformations, a complete list of extensions of the maximal Lie invariance algebra of equations from class* (1) *is exhausted by the cases* 1–5, $6|_{p=1}$, 7, 9, $10|_{\alpha=0}$, 11 *and* $13|_{\alpha=0}$.

3. Lie reduction and similarity solutions

The Lie symmetry operators found as a result of solving the group classification problem can be applied to construction of exact solutions of the corresponding equations. The method of reduction with respect to subalgebras of Lie invariance algebras is well known and quite algorithmic to use in most cases; we refer to the standard textbooks on the subject [47,49]. Cases 7–14 of Table 1 are presented by 'constant coefficient' nonlinear wave equations. Exact solutions of these equations have been already investigated intensively (see for example, [3,31,40]). That is why we choose cases 4–6 as representatives among 'truly' variable-coefficient nonlinear wave equations, which are most interesting for Lie reduction.

We first consider (1) with the fixed values of the parameter-functions $H = u^{-\frac{4}{3}}$, $k = k^1(x)$, i.e.

$$u_{tt} = \big(u^{-\frac{4}{3}}u_x\big)_x + k^1(x)u \tag{18}$$

where $k^1(x) = \epsilon \exp[\int \frac{q}{x^2+p}]$, $\epsilon = \pm 1$, $p \in \{-1, 0, 1\}$, $q \neq 0$. As shown in case 6 of Table 1, Eq. (18) admits the two-dimensional Lie invariance algebra $\mathfrak{g}$ generated by the operators

$$Q_1 = \partial_t, \qquad Q_2 = -2qt\partial_t + 4\big(x^2+p\big)\partial_x - 3(4x+q)u\partial_u,$$

which is a non-commutative algebra. A complete list of inequivalent non-zero subalgebras of $\mathfrak{g}$ is exhausted by the algebras $\langle Q_1\rangle$, $\langle Q_2\rangle$ and $\langle Q_1, Q_2\rangle$.

Table 2
Reduced ODEs and algebraic equation for Eq. (18).

N	Subalgebra	Ansatz	Reduced ODE
1	$\langle Q_1\rangle$	$u=(\varphi(\omega))^{-3}$, $\omega=x$	$3\varphi_{\omega\omega}=k^1(\omega)\varphi^{-3}$
2	$\langle Q_2\rangle$	$u=[(x^2+p)^{\frac{1}{2}}(k^1(x))^{\frac{1}{4}}\varphi(\omega)]^{-3}$ $\omega=t(k^1(x))^{\frac{1}{2}}$	$3q^2\omega^2\varphi_{\omega\omega}+\frac{9}{2}q^2\omega\varphi_\omega+12\varphi^{-5}\varphi_\omega^2-3\varphi^{-4}\varphi_{\omega\omega}$ $+\frac{3}{16}(q^2+16p)\varphi-\epsilon\varphi^{-3}=0$
3	$\langle Q_1,Q_2\rangle$	$u=C(x^2+p)^{\frac{3}{2}}(k^1(x))^{-\frac{3}{4}}$	$C^{\frac{4}{3}}=\frac{3}{16}(q^2+16p)$

Table 3
Reduced ODEs and algebraic equation for Eq. (19).

N	Subalgebra	Ansatz	Reduced ODE
1	$\langle Q_1\rangle$	$u=(\varphi(\omega))^{\frac{1}{\mu+1}}$, $\omega=x$ $u=\exp(\varphi(\omega))$, $\omega=x$	$\varphi_{\omega\omega}+\epsilon(\mu+1)\omega^q\varphi^{\frac{1}{\mu+1}}=0$ if $\mu\neq-1$ $\varphi_{\omega\omega}+\epsilon\omega^q e^{\varphi}=0$ if $\mu=-1$
2	$\langle Q_2\rangle$	$u=\lvert t\rvert^{-\frac{2(q+2)}{\mu q}}\varphi(\omega)$, $\omega=\lvert t\rvert^{\frac{2}{q}}x$	$(\varphi^\mu\varphi_\omega)_\omega+\epsilon\omega^q\varphi-\frac{2(q+2)}{\mu q}(\frac{2(q+2)}{\mu q}+1)\varphi$ $+\frac{2}{q}(\frac{4(q+2)}{\mu q}-\frac{2}{q}+1)\omega\varphi_\omega-\frac{4}{q^2}\omega^2\varphi_{\omega\omega}=0$
3	$\langle Q_1,Q_2\rangle$	$u=Cx^{\frac{q+2}{\mu}}$	$(q+2)(\mu q+\mu+q+2)C^{\mu+1}+\epsilon\mu^2C=0$

Table 4
Reduced ODEs and algebraic equation for Eq. (20).

N	Subalgebra	Ansatz	Reduced ODE
1	$\langle Q_1\rangle$	$u=(\varphi(\omega))^{\frac{1}{\mu+1}}$, $\omega=x$ $u=\exp(\varphi(\omega))$, $\omega=x$	$\varphi_{\omega\omega}+\epsilon(\mu+1)e^\omega\varphi^{\frac{1}{\mu+1}}=0$ if $\mu\neq-1$ $\varphi_{\omega\omega}+\epsilon e^{\varphi+\omega}=0$ if $\mu=-1$
2	$\langle Q_2\rangle$	$u=\lvert t\rvert^{-\frac{2}{\mu}}\varphi(\omega)$, $\omega=x+2\ln\lvert t\rvert$	$(\varphi^\mu\varphi_\omega)_\omega+\epsilon e^\omega\varphi-\frac{2}{\mu}(\frac{2}{\mu}+1)\varphi$ $+2(\frac{4}{\mu}+1)\varphi_\omega-4\varphi_{\omega\omega}=0$
3	$\langle Q_1,Q_2\rangle$	$u=Ce^{\frac{x}{\mu}}$	$(\mu+1)C^{\mu+1}+\epsilon\mu^2C=0$

Lie reduction of Eq. (18) to ordinary differential equations (ODEs) and an algebraic equation can be respectively made with the one-dimensional subalgebra $\langle Q_1\rangle$, $\langle Q_2\rangle$ and the two-dimensional subalgebra $\langle Q_1,Q_2\rangle$ which coincides with the whole algebra $\mathfrak{g}$. The associated ansatz and the reduced ODEs and algebraic equation are listed in Table 2.

Solving the algebraic equation we get $C=\pm\frac{3^{3/4}}{8}(q^2+16p)^{3/4}$. Thus we have the following exact solution

$$u=\pm\frac{3^{3/4}}{8}\left(q^2+16p\right)^{3/4}\left(x^2+p\right)^{\frac{3}{2}}\left(k^1(x)\right)^{-\frac{3}{4}}$$

of Eq. (18). The obtained reduced ODEs obviously have partial exact solutions which also lead to the above exact solution of Eq. (18) and can be constructed via reduction to algebraic equations. The problem is to find some different solutions. We are only able to reduce the order of the first ODE of Table 1. Namely, in the variables

$$y=\left(\omega^2+p\right)^{-1/2}\left(k^1(\omega)\right)^{-1/4}\varphi,\qquad \psi=\left(\omega^2+p\right)^{-1/2}\left(k^1(\omega)\right)^{-1/4}\left[\left(\omega^2+p\right)\varphi_\omega-\omega\varphi\right]$$

constructed with the induced symmetry operator $4(\omega^2+p)\partial_\omega+(4\omega+q)\varphi\partial_\psi$ the ODE takes the form $(4\psi-qy)\psi_y+q\psi+4py=\frac{4}{3}\epsilon y^{-3}$. A better way for construction of exact solutions for Eq. (18) with $p\leqslant0$ is to map them to equations for cases 4 and 5 of Table 1 with *additional equivalence transformations* and then study the latter cases.

The equations corresponding to cases 4 and 5 of Table 1 are

$$u_{tt}=\left(u^\mu u_x\right)_x+\epsilon x^q u,\tag{19}$$

$$u_{tt}=\left(u^\mu u_x\right)_x+\epsilon e^x u.\tag{20}$$

For each from these cases we denote the basis symmetry operators adduced in Table 1 by $\mathfrak{g}_1=\langle Q_1=\partial_t,\ Q_2=\mu qt\partial_t-2\mu x\partial_x-2(q+2)u\partial_u\rangle$ and $\mathfrak{g}_2=\langle Q_1=\partial_t,\ Q_2=\mu t\partial_t-2\mu\partial_x-2u\partial_u\rangle$. It is easy to know that structure and list of inequivalent subalgebras of the Lie invariance algebras of these two cases are the same as ones in Eq. (18). The associated ansatzes and reduced equations are listed in Tables 3 and 4.

Reduction to algebraic equations gives the following solutions of the initial equations (19) and (20) respectively:

$$u=\left[-\frac{q+2}{\epsilon\mu^2}(\mu q+\mu+q+2)\right]^{-\frac{1}{\mu}}x^{\frac{q+2}{\mu}};\qquad u=\left[-\frac{\mu+1}{\epsilon\mu^2}\right]^{-\frac{1}{\mu}}e^{\frac{x}{\mu}}.$$

Using these and the first two additional equivalence transformations in Table 1, i.e. $\tilde{t}=t$, $\tilde{x}=\frac{x-1}{x+1}$, $\tilde{u}=2^{-3/2}(x+1)^3u$ and $\tilde{t}=t$, $\tilde{x}=x^{-1}$, $\tilde{u}=x^3u$, we can obtain exact solutions of Eq. (18) with $p=-1$ and $p=0$, $q=-1$ respectively:

$$u=2\sqrt{2}\left[\frac{3(q+4)(q-2)}{32\epsilon}\right]^{\frac{3}{4}}(x-1)^{-\frac{3(q+4)}{8}}(x+1)^{\frac{3(q-4)}{8}};\qquad u=\left[\frac{3}{16\epsilon}\right]^{\frac{3}{4}}e^{-\frac{3}{4x}}x^{-3}.$$

Furthermore, some of the reduced ordinary differential equations in Tables 3 and 4 are the Emden–Fowler and the Lane–Emden equations and their different modifications [25,50]. For example, the first equation corresponding to case 1 of Table 3 are the standard Emden–Fowler equation, while the second one to case 1 of Table 3 is the generalized Lane–Emden equation. Solutions of these equations are known for a number of parameter values (see e.g. [25,50]). As a result, classes of exact solutions can be constructed for wave equations (19) and (20) for a wide set of the parameters μ and q. We omit these results in order to avoid a cumbersome enumeration.

4. On nonclassical symmetries

The notion of nonclassical symmetry (called also conditional symmetry) was introduced by Bluman and Cole in 1969 [10]. Their main appealing feature is that they yield solutions not obtainable from the classical Lie method. A precise and rigorous definition of this notion was suggested noticeably later [22,61] (see also [39] for a recent discussion on definition of nonclassical symmetry). Since then there is an explosion of research activity in the area of investigation of nonclassical symmetries. In a number of papers the reduction method with respect to nonclassical symmetry operators was successfully applied to obtain new non-Lie exact solutions of PDEs arising as models in different fields of physics, biology and chemistry [5,6,14,16,17,26,41,44–46]. Some of these works concern with nonlinear wave equations. See, for example, [21,31].

In what follows, we will study the conditional symmetries of Eqs. (1). In particular, the conditional symmetries of the special cases $H(u)=u$, $k(x)=1$, i.e., nonlinear wave equations

$$u_{tt}=(uu_x)_x+u \tag{21}$$

are given. From Table 1, it is easy to know that Eq. (21) (case 9) admits three-dimensional Lie algebra $\mathfrak{g}$ of its infinitesimal Lie symmetries with a basis:

$$X_1=\partial_x,\qquad X_2=x\partial_x+2u\partial_u,\qquad X_3=\partial_t. \tag{22}$$

The corresponding one-parameter groups are space translations and scale transformations.

To discuss the conditional symmetries of Eq. (21), we will first treat equation (1) from a geometric point of view as a hypersurface E in the space J^2 of 2-jets of local functions $f(t,x)$ defined on the space R^2 of independent variables t, x [47]. Let

$$Q=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u,\qquad (\tau,\xi)\neq(0,0) \tag{23}$$

be an operator on the space $R^2\times R^1$ with coordinates t, x, and u. Then all functions invariant under Q and only such functions satisfy a first order differential equation

$$\tau u_t+\xi u_x-\eta=0 \tag{24}$$

called the invariant surface condition $E_Q^{(1)}$.

Denote by $E_Q^{(2)}\subset J^2$ the first prolongation of $E_Q^{(1)}$. The system $E_Q^{(2)}$ consists of (24) and the equations obtained by t- and x-differentiation of (24). Denote also by $Q_{(2)}$ the second prolongation of Q to the space J^2.

Definition 1. The differential equation (1) is called conditionally invariant with respect to the operator Q if the relation

$$Q_{(2)}\big[u_{tt}-\big(H(u)u_x\big)_x-k(x)u\big]\big|_{E\cap E_Q^{(2)}}=0 \tag{25}$$

holds, which is called the conditional (or nonclassical) invariance criterion. Then Q is called conditional symmetry (or nonclassical symmetry) of Eq. (1).

In general the set of all conditional symmetries does not form a Lie algebra. This is due to the fact that (25) is a nonlinear system. However, if Q is a conditional symmetry of (1), so is $f(t,x,u)Q$ for any smooth function f. This last property reduces the search of conditional symmetries to the following two cases up to the usual equivalence of reduction operators:

1. Operator with $\tau=0$, $\xi=1$ (conditional symmetry of the first type);
2. Operator with $\tau=1$ (conditional symmetry of the second type).

In what follows, we discuss these two cases in details.

Case I: We begin by considering the conditional symmetry operator of the first kind:

$$Q = \partial_x + \eta(t, x, u)\partial_u. \tag{26}$$

With the assumptions $\tau = 0$, $\xi = 1$ the determining equations (25) are as follows

$$\begin{aligned} &\eta_{tu} = 0, \qquad \eta_{uu} = 0, \\ &\eta_{tt} + H_u\left(-3\eta\eta_x - 2\eta^2\eta_u\right) - H\left(\eta_{xx} + 2\eta_{xu}\eta\right) - \eta^3 H_{uu} + \eta_u ku - k_x u - k\eta = 0. \end{aligned} \tag{27}$$

From the first two equations we obtain that

$$\eta(t, x, u) = A(x)u + B(t, x). \tag{28}$$

Substituting the latest equation with $H(u) = u$ into the last equation of system (27), we can see that the functions $A(x)$ and $B(t, x)$ satisfy the overdetermined system:

$$\begin{aligned} &A_{xx} + 5AA_x + 2A^3 = 0, \\ &B_{xx} + 3AB_x + 4A^2B + k_x + 5A_xB = 0, \\ &B_{tt} - 3BB_x - (2AB + k)B = 0. \end{aligned} \tag{29}$$

The last two equations of system (29) imply the compatibility condition

$$14AB_x^2 + \left(36A^2B + 46A_xB + 7k_x\right)B_x - \left(37AA_x + 18A^3\right)B^2 + (Ak_x + 2k_{xx})B + kk_x = 0 \tag{30}$$

obtained by cross-differential. The second equation of (29) is a differential consequence of (30) provided the equation

$$\begin{aligned} &\left(60A_x - 48A^2\right)B_x^2 - \left(348A^3B + 510ABA_x + 48Ak_x - 9k_{xx}\right)B_x - \left(267A_x^2 + 233A^2A_x + 70A^4\right)B^2 \\ &\quad - \left(64A^2k_x - 2k_{xxx} + 80A_xk_x - Ak_{xx}\right)B + kk_{xx} - 6k_x^2 = 0 \end{aligned} \tag{31}$$

is satisfied.

Suppose that $k(x) = 1$ and eliminate B_x from (30) and (31) we come to the equation

$$B^3\left(2A^2 + A_x\right)\left(302707A^6 + 1616526A^4A_x + 2869917A^2A_x^2 + 1694916A_x^3\right) = 0. \tag{32}$$

If we take the third factor in (32) and the first equation for the function $A(x)$ in system (29), then that overdetermined system for the function $A(x)$ admits the unique solution $A(x) = 0$. Hence, solving system (29) with $k(x) = 1$ and $A(x) = 0$, we can obtain

$$B(t, x) = a(t)x + b(t),$$

where the functions $a(t)$ and $b(t)$ satisfy the system of ODE:

$$a_{tt} - 3a^2 - a = 0, \qquad b_{tt} - 3ab - b = 0. \tag{33}$$

The general solution of the first equation of this system can be expressed in terms of Weierstrass functions and then the second equation of system (33) is nothing but a Lame equation. Hence, the solutions of this system can be given explicitly. Here we provide only some particular solutions for interest:

$$\begin{aligned} &a(t) = -\frac{1}{3}, \qquad b(t) = c_1t + c_2, \\ &a(t) = -\frac{1}{2\cosh^2(\frac{1}{2}t)}, \qquad b(t) = c_2\left[8\tanh\left(\frac{1}{2}t\right) + 2\cosh t\tanh\left(\frac{1}{2}t\right) + \frac{3t}{\cosh(\frac{1}{2}t)}\right] + \frac{c_1}{4\cosh^2(\frac{1}{2}t)}, \\ &a(t) = \frac{1}{2\sinh^2(\frac{1}{2}t)}, \qquad b(t) = c_2\left[-8\coth\left(\frac{1}{2}t\right) + 2\cosh t\coth\left(\frac{1}{2}t\right) + \frac{3t}{\sinh(\frac{1}{2}t)}\right] + \frac{c_1}{4\sinh^2(\frac{1}{2}t)}. \end{aligned} \tag{34}$$

For the first solution of (34) yield an exact explicit solution of Eq. (21) obtainable by solving equation (24), which is an ordinary differential equation in the variable x for the symmetries of second type, and subsequent solution of Eq. (21) for the 'constants' of integration actually depending on the variable t:

$$u(t, x) = -\frac{1}{6}\left(9t^2 + 27\right)c_1^2 + t(x - 3c_2)c_1 - \frac{1}{6}(x - 3c_2)^2 + c_3e^{-\frac{\sqrt{6}}{3}t} + c_4e^{-\frac{\sqrt{6}}{3}t}$$

where c_i are parameters. Similarly, we can obtain another two solutions of (21) corresponding to the last two cases of system (34). Because of these solutions are very complicated, we only list the solutions of (21) which results from Eq. (24) and system (34) with $c_2 = 0$:

$$u(t,x) = -\frac{e^t x(-c_1+x)}{1+e^{2t}+2e^t} + \frac{(3+e^{-t})c_3}{e^t+1} + \frac{(3e^t+e^{2t})c_4}{e^t+1} + \frac{(4e^t+1){c_1}^2e^{-t}}{12e^{2t}+24e^t+12};$$
$$u(t,x) = -\frac{e^t x(c_1+x)}{-1-e^{2t}+2e^t} - \frac{(-3+e^{-t})c_3}{e^t-1} + \frac{(-3e^t+e^{2t})c_4}{e^t-1} - \frac{(4e^t-1){c_1}^2e^{-t}}{-12e^{2t}+24e^t-12}.$$

If we set $B(t,x) = 0$ and $k(x) = 1$, then last two equations of (29) are satisfied and we arrive at an infinitesimal conditional symmetry

$$Q = \partial_x + A(x)\partial_u$$

with the function $A(x)$ satisfying the ODE $A_{xx} + 5AA_x + 2A^3 = 0$. Particular solution $A(x) = 2/x$ of the latter equation yields exact solutions of the nonlinear wave equation

$$u(t,x) = w(t)x^2$$

with $w(t)$ satisfying

$$w_{tt} - 6w^2 - w = 0,$$

while particular solution $A(x) = 1/(2x)$ yields the exact solution

$$u(t,x) = \left(c_1e^{-t} + c_2e^t\right)\sqrt{x}.$$

Finally, if we take the second factor $A_x + 2A^2 = 0$ in (32), which implies the first equation of (29), we arrive at an infinitesimal conditional symmetry

$$Q = \partial_x + \left(\frac{u}{2x} + \varphi(t)x\right)\partial_u$$

where the function $\varphi(t)$ satisfies the ODE

$$\varphi_{tt} - 4\varphi^2 - \varphi = 0.$$

If we take the particular solution

$$\varphi(t) = -1/4, \qquad \varphi(t) = -\frac{3}{8\cosh^2(\frac{1}{2}t)}, \qquad \varphi(t) = \frac{3}{8\sinh^2(\frac{1}{2}t)} \tag{35}$$

we obtain the exact solutions

$$u(t,x) = -1/6x^2 + \sqrt{x}c_1e^{1/4\sqrt{6}t} + \sqrt{x}c_2e^{-1/4\sqrt{6}t};$$
$$u(t,x) = -\frac{x^2e^t}{e^{2t}+2e^t+1} + \sqrt{x}\left(c_1\,\mathit{Legendre\,P}\left(3/2, 2, \frac{e^t-1}{e^t+1}\right) + c_2\,\mathit{Legendre\,Q}\left(3/2, 2, \frac{e^t-1}{e^t+1}\right)\right);$$
$$u(t,x) = -\frac{x^2e^t}{-e^{2t}+2e^t-1} + \sqrt{x}\left(c_1\,\mathit{Legendre\,P}\left(3/2, 2, \frac{e^t+1}{e^t-1}\right) + c_2\,\mathit{Legendre\,Q}\left(3/2, 2, \frac{e^t+1}{e^t-1}\right)\right)$$

of Eq. (21), where $\mathit{Legendre\,P}(v,u,t)$ and $\mathit{Legendre\,Q}(v,u,t)$ are the associated Legendre functions of the first kind and the second kind respectively.

Case II: Now we consider the second kind conditional symmetry operator:

$$Q = \partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{36}$$

For operator (36) it is possible to eliminate the derivatives u_t, u_{tt}, u_{tx}, and u_{xx} from (25) by solving the system $E \cap E_Q^{(2)}$ with respect to these derivatives and by substituting the expressions obtained into (25). After elimination, Eq. (25) becomes a third-order polynomial in the derivative u_x. Setting the coefficients of the polynomial equal to zero yields the determining equations for the functions ξ and η:

$$2\left(\frac{\xi H_u}{H-\xi^2}-\xi_u\right)\xi\xi_u-\frac{2\xi_u H H_u}{H-\xi^2}-\xi_u H_u+H\xi_{uu}+2H_u\xi_u-\xi^2\xi_{uu}=0;$$

$$-H\eta_{uu}+\frac{2\xi_u H(\xi\xi_x-\xi_t)}{H-\xi^2}+2H\xi_{xu}-\eta H_{uu}+\frac{\eta H_u^2}{H-\xi^2}+2H_u\xi_x-\frac{2HH_u\xi_x}{H-\xi^2}+2\left[\eta_u-\xi_x-\frac{\xi(\xi\xi_x-\xi_t)}{H-\xi^2}\right]\xi_u\xi$$
$$-2\left(\frac{\xi H_u}{H-\xi^2}-\xi_u\right)\eta\xi_u-2H_u\eta_u+\xi^2\eta_{uu}+2\xi_{uu}\eta\xi+2\xi_{tu}\xi-2\left(\frac{\xi H_u}{H-\xi^2}-\xi_u\right)\xi_t+\eta_u H_u=0;$$

$$-ku\xi_u+H\xi_{xx}-\frac{\eta H_u(\xi\xi_x-\xi_t)}{H-\xi^2}+\frac{2\xi_u H(\eta_t-\xi\eta_x-ku)}{H-\xi^2}-2H\eta_{xu}-\xi_{uu}\eta^2-2\xi_{tu}\eta-2\eta_{tu}\xi$$
$$-\xi_{tt}-2\left[\eta_u-\xi_x-\frac{\xi(\xi\xi_x-\xi_t)}{H-\xi^2}\right]\xi_t+\frac{2H(\xi\xi_x-\xi_t)\xi_x}{H-\xi^2}+2\left[\eta_x-\frac{\xi(\eta_t-\xi\eta_x-ku)}{H-\xi^2}\right]\xi_u\xi$$
$$-2\left[\eta_u-\xi_x-\frac{\xi(\xi\xi_x-\xi_t)}{H-\xi^2}\right]\xi_u\eta-2H_u\eta_x-2\eta_{uu}\eta\xi=0;$$

$$-\frac{\eta H_u(\eta_t-\xi\eta_x-ku)}{H-\xi^2}-k\eta-2\left[\eta_x-\frac{\xi(\eta_t-\xi\eta_x-ku)}{H-\xi^2}\right]\xi_t+2\frac{H(\eta_t-\xi\eta_x-ku)\xi_x}{H-\xi^2}$$
$$-2\left[\eta_x-\frac{\xi(\eta_t-\xi\eta_x-ku)}{H-\xi^2}\right]\xi_u\eta+2\eta\eta_{tu}-\xi k_x u+\eta_{uu}\eta^2+\eta_{tt}+ku\eta_u-H\eta_{xx}=0.$$

Substituting $H(u)=u$ and $k(x)=1$ into above system, we get

$$\xi^4\xi_{uu}+2\xi^3\xi_u^2-2u\xi^2\xi_{uu}-2u\xi\xi_u^2+u^2\xi_{uu}+\xi^2\xi_u-u\xi_u=0;$$
$$-\xi^4\eta_{uu}-2\xi^3\eta\xi_{uu}-2\xi^3\eta_u\xi_u-2\xi^2\eta\xi_u^2-2u\xi^2\xi_{xu}+2u\xi^2\eta_{uu}+2u\xi\eta\xi_{uu}+2u\xi\eta_u\xi_u$$
$$+2u\eta\xi_u^2-2\xi^3\xi_{tu}+2u^2\xi_{xu}-u^2\eta_{uu}+2u\xi\xi_{tu}+\xi^2\eta_u-2\xi^2\xi_x-2\xi\eta\xi_u-u\eta_u-2\xi\xi_t+\eta=0;$$
$$\xi_{uu}\eta^2\xi^2+2\eta_{uu}\eta\xi^3+2\xi^2\eta\eta_u\xi_u+3u\xi^2\xi_u+2\eta_{tu}\xi^3-u\xi_{uu}\eta^2-2u\eta_{uu}\eta\xi+2u\eta_{xu}\xi^2$$
$$+2\xi_{tu}\eta\xi^2-u\xi_{xx}\xi^2-2\eta_t\xi^2\xi_u-2u\eta_x\xi\xi_u+2\xi_x\xi^2\xi_t+2u\xi_x^2\xi-2\xi_t\eta\xi\xi_u+2u\eta\xi_x\xi_u$$
$$-2u\eta\eta_u\xi_u-2\xi^2\xi_x\xi_t+2\xi^2\eta_u\xi_t+2u\xi\eta_x\xi_u-3u^2\xi_u-2u\xi\eta_{tu}+\xi^2\xi_{tt}-2u^2\eta_{xu}$$
$$-2u\eta\xi_{tu}+u^2\xi_{xx}+2u\eta_t\xi_u-\xi_x\eta\xi-2\xi_t^2\xi+2\xi^2\eta_x-2u\eta_u\xi_t-u\xi_{tt}+\eta\xi_t-2u\eta_x=0;$$
$$-2\eta_x\eta\xi^2\xi_u-u\xi^2\eta_u-2\xi^2\eta\eta_{tu}+u\eta^2\eta_{uu}-\xi^2\eta^2\eta_{uu}+2\xi\xi_u\eta\eta_t-2\eta_x\xi^2\xi_t-2u\eta_x\xi\xi_x$$
$$+u\xi^2\eta_{xx}+\xi^2\eta-2u\eta\eta_x\xi_u+2\xi^2\eta_x\xi_t+2\xi^2\eta\eta_x\xi_u-2u\xi\eta\xi_u+u^2\eta_u+2u\eta\eta_{tu}-\xi^2\eta_{tt}$$
$$-u^2\eta_{xx}+2\eta_t\xi\xi_t+2u\eta_t\xi_x+\xi\eta\eta_x-2u\eta_x\xi_t+u\eta_{tt}-\eta\eta_t-2u\xi\xi_t-2u^2\xi_x=0. \tag{37}$$

Since these four equations are very complicated, we restrict our analysis to infinitesimal conditional symmetries

$$Q=\partial_t+\xi(t,x)\partial_x+\big(f(t,x)u+g(t,x)\big)\partial_u \tag{38}$$

with special dependence on the variable u admissible by Eq. (21). Eqs. (37) imply the following relations:

$$f_x=0,\qquad \xi_{xx}=0,\qquad g=-f\xi^2+2\xi^2\xi_x+2\xi\xi_t.$$

Hence,

$$f(x)=a(t),\qquad \xi(t,x)=b(t)x+c(t) \tag{39}$$

and the problem of obtaining the infinitesimal conditional symmetries (38) reduces to solving a system of sixteen nonlinear ordinary differential equations of the third order for the unknown functions $a(t)$, $b(t)$ and $c(t)$, which is a consequence of (37). Due to nonlinearity of this system, we find it is difficult to be solved explicitly. So we can make a further restriction for the functions $a(t)$, $b(t)$ and $c(t)$. We can set the function $c(t)=0$ or $b(t)=0$ with no loss of generality. Therefore, we will consider the vector fields of conditional symmetries

$$Q=\partial_t+bx\partial_x+\big(au+x^2\big(2b^3+2bb_t-ab^2\big)\big)\partial_u \tag{40}$$

and

$$Q=\partial_t+cx\partial_x+\big(au+2cc_t-ac^2\big)\partial_u. \tag{41}$$

Now, we could continue our analysis by trying to obtain particular solutions of ordinary differential systems. Instead, we prefer a more systematic method of using the symmetry properties of infinitesimal conditional symmetries with respect to the classical Lie symmetries [60].

Suppose that X is an infinitesimal classical symmetry of Eq. (21). Denote by $\exp(\tau X)$ the one-parameter transformation group on the space R^3 of the points (t,x,u) associated with X. This group generates the induced actions $\exp(\tau X)^*$ on the space $C^\infty(R^3)$ of smooth functions and $\exp(\tau X)_*$ on the space $D(R^3)$ of symmetry operator on R^3:

$$\begin{aligned}&\exp(\tau X)^*(F)(t,x,u) = F\circ\exp(\tau X)(t,x,u),\\&\exp(\tau X)_*(Q) = \exp(-\tau X)^*\circ Q\circ\exp(\tau X)^*\end{aligned}\tag{42}$$

where $F(t,x,u)$ is a smooth function, Q is a vector field on R^3 considered as a first order linear differential operator on $C^\infty(R^3)$. It was demonstrated in Theorems 4 and 6 of [60] that if Q is an infinitesimal conditional symmetry of a differential equation and X is an infinitesimal classical symmetry of the same equation, then $\exp(\tau X)_*(Q)$ is also an infinitesimal symmetry.

The infinitesimal conditional symmetries Q invariant under the vector field X must satisfy the commutation relation

$$[X,Q] = \lambda(t,x,u)Q.\tag{43}$$

Similarly to invariant solutions of differential equations, the invariant vector fields of conditional symmetries break up into conjugacy classes. Vector fields belonging to the same class can be obtained from a particular vector field of the class by transformations (42). Therefore, it is sufficient to obtain the vector fields invariant under the representatives of the conjugacy classes of the Lie algebra $\mathfrak{g}$ of the classical symmetries with respect to inner automorphisms. The standard methods of obtaining conjugacy classes of subalgebras under inner automorphisms [49] lead to the following list of representatives of conjugacy classes (optimal subalgebras) of the Lie algebra $\mathfrak{g}$ with the basis (22):

$$\mathfrak{g}_1 = L(X_1),\qquad \mathfrak{g}_2 = L(X_3),\qquad \mathfrak{g}_3 = L(X_1 - X_3),\qquad \mathfrak{g}_4 = L(X_1 + X_3),\qquad \mathfrak{g}_5 = L(X_2 - \alpha X_3)\tag{44}$$

where $L(X_i)$ denotes the linear span of X_i, $\alpha = const$. It is evident that the vector fields (38) invariant under $X_1 = \partial_x$ must look like

$$Q = \partial_t + \psi(t)u\partial_u\tag{45}$$

by calculating the commutator of vector fields X_1 and Q. Substituting (45) into determining equations (37), we arrive at the ordinary differential equation for the function $\psi(t)$:

$$\psi_{tt} + \psi\psi_t + \psi = 0.\tag{46}$$

However, for the infinitesimal conditional symmetries (40) and (41) invariant under $X_3 = \partial_t$, $X_1 \pm X_3 = \partial_t \pm \partial_x$, $X_2 - \alpha X_3 = x\partial_x + 2u\partial_u - \alpha\partial_t$ the functions $a(t)$, $b(t)$ and $a(t)$, $c(t)$ are constants. Substituting these constants into determining Eqs. (37), we know that they are all zero. Hence, there exist no infinitesimal conditional symmetries at all for these four subalgebras.

From the discussion in cases I and II, we can arrive at

Theorem 5. *Eq.* (1) *with* $H(u) = u$ *and* $k(x) = 1$ *is conditionally invariant under the following operators*:

(1) $Q = \partial_x + 2/x\partial_u$;
(2) $Q = \partial_x + 1/(2x)\partial_u$;
(3) $Q = \partial_x + [a(t)x + b(t)]\partial_u$;
(4) $Q = \partial_x + (\frac{u}{2x} + \varphi(t)x)\partial_u$;
(5) $Q = \partial_t + \psi(t)u\partial_u$,

where $a(t)$, $b(t)$, $\varphi(t)$ *and* $\psi(t)$ *are given by Eqs.* (34), (35) *and* (46).

Remark 4. It should be noted that the partition of the reduction operators (23) for wave equations (1) into the above two cases (i.e., $\tau = 0$, $\xi = 1$ and $\tau = 1$) up to the usual equivalence is not natural (see [38] for details). In fact, according to the discussion of singular reduction operators of wave type equations in Section 6 in reference [38] by Popovych et al., we can obtain two different cases of inequivalent reduction operators with respect to the usually equivalence relation: the singular case $\tau = 1/\sqrt{H}$, $\xi = 1$ and the regular case $\tau \neq \pm 1/\sqrt{H}$ and $\xi = 1$. In this way, the conditional symmetries presented in cases 1 and 2 are all particular regular cases, while for the singular reduction operators of Eq. (1), it will be investigated in another publication.

5. Conclusion and remarks

In this paper we present a complete group classification of the generalized quasi-linear wave equations (1) by using the Lie–Ovsiannikov method, additional equivalent transformation and furcate split method. The main results on classification are collected in Table 1 where we list inequivalent cases of extensions with the corresponding Lie invariance algebras. For a number of *truly* 'variable coefficient' wave equations from the list we construct optimal systems of inequivalent subalgebras, corresponding Lie ansätze and exact solutions which are presented in Tables 2–4. Nonclassical symmetries of the class of generalized quasi-linear wave equations (1) with $H(u) = u$, $k(x) = 1$ are also investigated. This enabled to obtain some exact solutions which are invariant under certain conditional symmetries.

The present paper is a preliminary group analysis of the two classes of hyperbolic type nonlinear partial differential equations (2) and (3). Therefore, further investigations of different properties such as group and conservation law classifications, potential symmetries, conditional symmetries and exact solutions as well as physical application of these two classes of equations would be extremely interesting. These results will be reported in subsequent publications.

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References

[1] I.Sh. Akhatov, R.K. Gazizov, N.Kh. Ibragimov, Nonlocal symmetries: A heuristic approach, J. Soviet Math. 55 (1991) 1401–1450; Itogi Nauki i Tekhniki, Current Problems in Mathematics, Newest Results, vol. 34, 1989, pp. 3–83 (in Russian).
[2] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic, New York, 1972, pp. 50–52.
[3] W.F. Ames, E. Adams, R.J. Lohner, Group properties of $u_{tt} = [f(u)u_x]_x$, Internat. J. Non-Linear Mech. 16 (1981) 439–447.
[4] D.J. Arrigo, Group properties of $u_{xx} - u_y^m u_{yy} = f(u)$, Internat. J. Non-Linear Mech. 26 (1991) 619–629.
[5] D.J. Arrigo, J.M. Goard, P. Broadbridge, Nonclassical solutions are non-existent for the heat equation and rare for nonlinear diffusion, J. Math. Anal. Appl. 202 (1996) 259–279.
[6] D.J. Arrigo, J.M. Hill, P. Broadbridge, Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, IMA J. Appl. Math. 52 (1994) 1–24.
[7] A. Bamberger, G. Chavent, P. Lailly, About the stability of the inverse problem in 1-D wave equations: Applications to the interpretation of seismic profiles, Appl. Math. Optim. 5 (1979) 1–47.
[8] P. Baldi, M. Berti, Forced vibrations of a nonhomogeneous string, SIAM J. Math. Anal. 40 (2008) 382–412.
[9] G.W. Bluman, A.F. Cheviakov, Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation, J. Math. Anal. Appl. 333 (2007) 93–111.
[10] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.
[11] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, New York, 1989.
[12] G.W. Bluman, Temuerchaolu, R. Sahadevan, Local and nonlocal symmetries for nonlinear telegraph equation, J. Math. Phys. 46 (2005) 023505.
[13] V.M. Boyko, V.O. Popovych, Group classification of Galilei-invariant higher-orders equations, Proc. Inst. Math. Natl. Acad. Sci. Ukraine 36 (2001) 45–50.
[14] B.H. Bradshaw-Hajek, M.P. Edwards, P. Broadbridge, G.H. Williams, Nonclassical symmetry solutions for reaction–diffusion equations with explicit spatial dependence, Nonlinear Anal. 67 (2007) 2541–2552.
[15] S.C. Chikwendu, Non-linear wave propagation solutions by Fourier transform perturbation, Internat. J. Non-Linear Mech. 16 (1981) 117–128.
[16] P.A. Clarkson, E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Phys. D 70 (1994) 250–288.
[17] P.A. Clarkson, P. Winternitz, Nonclassical symmetry reductions for the Kadomtsev–Petviashvili equation, Phys. D 49 (1991) 257–272.
[18] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience Publishers, New York, 1953, p. 291.
[19] A. Donato, Similarity analysis and nonlinear wave propagation, Internat. J. Non-Linear Mech. 22 (1987) 307–314.
[20] P.G. Estévez, C.Z. Qu, Separation of variables in nonlinear wave equations with variable wave speed, Theoret. Math. Phys. 133 (2002) 1490–1497.
[21] M.V. Foursov, E.M. Vorob'ev, Solutions of the nonlinear wave equation $u_{tt} = (uu_x)_x$ invariant under conditional symmetries, J. Phys. A 29 (1996) 6363–6373.
[22] W.I. Fushchych, I.M. Tsyfra, On a reduction and solutions of the nonlinear wave equations with broken symmetry, J. Phys. A 20 (1987) L45–L48.
[23] W.I. Fushchych, W.M. Shtelen, N.I. Serov, Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics, Kluwer, Dordrecht, 1993 (in English).
[24] M.L. Gandarias, M. Torrisi, A. Valenti, Symmetry classification and optimal systems of a non-linear wave equation, Internat. J. Non-Linear Mech. 39 (2004) 389–398.
[25] H. Goenner, P. Havas, Exact solutions of the generalized Lane–Emden equation, J. Math. Phys. 41 (2000) 7029–7042.
[26] A.M. Grundland, J. Tafel, On the existence of nonclassical symmetries of partial differential equations, J. Math. Phys. 36 (1995) 1426–1434.
[27] J.Y. Hu , C.Z. Qu, Functionally separable solutions to nonlinear wave equations by group foliation method, J. Math. Anal. Appl. 330 (2007) 298–311.
[28] D.J. Huang, N.M. Ivanova, Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations, J. Math. Phys. 48 (2007) 073507, 23 page.
[29] D.J. Huang, J.Q. Mei, H.Q. Zhang, Group classification and exact solutions of a class of variable coefficient nonlinear wave equations, Chin. Phys. Lett. 26 (2009) 050202.
[30] D.J. Huang, S.G. Zhou, J.Q. Mei, H.Q. Zhang, Lie reduction and conditional symmetries of some variable coefficient nonlinear wave equations, Commun. Theor. Phys. 53 (2010) 1–5.
[31] N.H. Ibragimov (Ed.), Lie Group Analysis of Differential Equations – Symmetries, Exact Solutions and Conservation Laws, vol. 1, CRC Press, Boca Raton, FL, 1994.
[32] N.H. Ibragimov, M. Torrisi, A. Valenti, Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, J. Math. Phys. 32 (1991) 2988–2995.
[33] N.M. Ivanova, R.O. Popovych, C. Sophocleous, Conservation laws of variable coefficient diffusion–convection equations, in: Proceedings of Tenth International Conference in Modern Group Analysis, Larnaca, Cyprus, 2004, pp. 107–113.
[34] N.M. Ivanova, R.O. Popovych, C. Sophocleous, Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification, arXiv:0710.2731 [math-ph].
[35] N.M. Ivanova, C. Sophocleous, On the group classification of variable coefficient nonlinear diffusion–convection equations, J. Comput. Appl. Math. 197 (2006) 322–344.
[36] J.G. Kingston, C. Sophocleous, On form-preserving point transformations of partial differential equations, J. Phys. A 31 (1998) 1597–1619.

[37] J.G. Kingston, C. Sophocleous, Symmetries and form-preserving transformations of one-dimensional wave equations with dissipation, Internat. J. Non-Linear Mech. 36 (2001) 987–997.
[38] M. Kunzinger, R.O. Popovych, Singular reduction operators in two dimensions, J. Phys. A 41 (2008) 505201, 24 pp., arXiv:0808.3577.
[39] M. Kunzinger, R.O. Popovych, Is a nonclassical symmetry a symmetry, in: Proceedings of 4th Workshop Group Analysis of Differential Equations and Integrability, 2009, arXiv:0903.0821.
[40] V. Lahno, R. Zhdanov, O. Magda, Group classification and exact solutions of nonlinear wave equations, Acta Appl. Math. 91 (2006) 253–313.
[41] D. Levi, P. Winternitz, Non-classical symmetry reduction: Example of the Boussinesq equation, J. Phys. A 22 (1989) 2915–2924.
[42] S.V. Meleshko, Group classification of equations of two-dimensional gas motions, Prikl. Mat. Mekh. 58 (1994) 56–62 (in Russian); J. Appl. Math. Mech. 58 (1994) 629–635.
[43] A.G. Nikitin, R.O. Popovych, Group classification of nonlinear Schrödinger equations, Ukrainian Math. J. 53 (2001) 1053–1060.
[44] M.C. Nucci, Nonclassical symmetries as special solutions of heir-equations, J. Math. Anal. Appl. 279 (2003) 168–179.
[45] M.C. Nucci, P.A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions: An example of the Fitzhugh–Nagumo equation, Phys. Lett. A 164 (1992) 49–56.
[46] M.C. Nucci, P.G.L. Leach, The determination of nonlocal symmetries by the technique of reduction of order, J. Math. Anal. Appl. 251 (2000) 871–884.
[47] P.J. Olver, Application of Lie Groups to Differential Equations, Springer, New York, 1986.
[48] A. Oron, P. Rosenau, Some symmetries of the nonlinear heat and wave equations, Phys. Lett. A 118 (1986) 172–176.
[49] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.
[50] A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, Chapman Hall/CRC, Boca Raton, 2003.
[51] R.O. Popovych, R.M. Cherniha, Complete classification of Lie symmetries of systems of two-dimensional Laplace equations, Proc. Inst. Math. Natl. Acad. Sci. Ukraine 36 (2001) 212–221.
[52] R.O. Popovych, N.M. Ivanova, New results on group classification of nonlinear diffusion–convection equations, J. Phys. A 37 (2004) 7547–7565, math-ph/0306035.
[53] E. Pucci, Group analysis of the equation $u_{tt} + \lambda u_{xx} = g(u, u_x)$, Riv. Mat. Univ. Parma 12 (4) (1987) 71–87.
[54] E. Pucci, M.C. Salvatori, Group properties of a class of semilinear hyperbolic equations, Internat. J. Non-Linear Mech. 21 (1986) 147–155.
[55] C. Sophocleous, J.G. Kingston, Cyclic symmetries of one-dimensional non-linear wave equations, Internat. J. Non-Linear Mech. 34 (1999) 531–543.
[56] E.S. Suhubi, A. Bakkaloglu, Group properties and similarity solutions for a quasi-linear wave equation in the plane, Internat. J. Non-Linear Mech. 26 (1991) 567–584.
[57] M. Torrisi, A. Valenti, Group properties and invariant solutions for infinitesimal transformations of a nonlinear wave equation, Internat. J. Non-Linear Mech. 20 (1985) 135–144.
[58] O.O. Vaneeva, A.G. Johnpillai, R.O. Popovych, C. Sophocleous, Group analysis of nonlinear fin equations, Appl. Math. Lett. 21 (2008) 248–253.
[59] O.F. Vasilenko, I.A. Yehorchenko, Group classification of multidimensional nonlinear wave equations, Proc. Inst. Math. Natl. Acad. Sci. Ukraine 36 (2001) 63–66.
[60] E.M. Vorob'ev, Symmetries of compatibility conditions for systems of differential equations, Acta Appl. Math. 26 (1992) 61–86.
[61] R.Z. Zhdanov, I.M. Tsyfra, R.O. Popovych, A precise definition of reduction of partial differential equations, J. Math. Anal. Appl. 238 (1999) 101–123, math-ph/0207023.